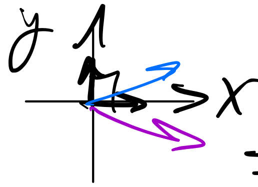
4.5 – Coordinates and Basis

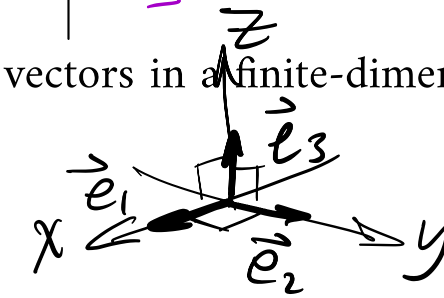


Due Fri

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a **basis** for V if:

a) S spans V. 4.3

b) S is linearly independent. 4.4



$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are the standard basis vectors for R^3. These relate to mutually perpendicular axes. However, any 3 vectors in R^3 that do not lie in the same plane can be used to represent any other vector in R^3. These 3 vectors are called basis vectors.

#1 Use the determinant of a coefficient matrix to show that the following set of vectors forms a basis for R^2: $\{(2,\ 1),\ (3,\ 0)\}$.

- Spans R^2
- Linearly independent

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$. If $A\vec{x} = \vec{b}$ always has a solution, then the set spans R^2. If $A\vec{x} = \vec{0}$ has only the trivial solution, then the set is independent. We use det(A) because of Equivalent Statements. Here, $\det(A) = -3 \neq 0$, so the set is a basis for R^2.

#8 Show that the following vectors do not form a basis for P_2.

$1 - 3x + 2x^2$, $1 + x + 4x^2$, $1 - 7x$

$$\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} \begin{matrix} 1 & 1 \\ -3 & 1 \\ 2 & 4 \end{matrix}$$

$2 - 28 + 0$

$-26 - (-26) = 0$

$0 - 14 - 12$

the set is not a basis for P_2.

Definition: A basis in which the listed order of the vectors matters is called an **ordered basis**.

P_2:

$1 = 1 + 0x + 0x^2$

$x = 0 + x + 0x^2$

$x^2 = 0 + 0x + x^2$

$\{(1,0,0), (0,1,0), (0,0,1)\}$

Some Standard bases

$\{\hat{i},\ \hat{j}\}$ is a basis for R^2 (this is the same as $\{\mathbf{e}_1, \mathbf{e}_2\}$).

$\{\hat{i},\ \hat{j},\ \hat{k}\}$ is basis for R^3 (this is the same as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$).

$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for R^n.

$\{1,\ x,\ x^2,\ \dots,\ x^n\}$ is a basis for P_n.

$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis for M_{22}.

The basis for the zero vector space $V = \{\vec{0}\}$ is $\emptyset$ (the empty set)

Theorem 4.5.1 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + \dots + c_n\,\mathbf{v}_n$ in exactly one way.

$\Longrightarrow$

- Spans
- lin. indep.

always exists and is unique

Pf: Let $\vec{v} \in V$. Since S spans V, $\exists$ $c_1, c_2, \ldots c_n \ni \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

Suppose $\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$.

$\vec{v} - \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n - (k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n)$

$\Rightarrow \vec{0} = (c_1 - k_1)\vec{v}_1 + (c_2 - k_2)\vec{v}_2 + \cdots + (c_n - k_n)\vec{v}_n$

Since S is lin. indep., $c_i - k_i = 0 \ \forall i$

$\Rightarrow c_i = k_i$. Thus the representation is unique.

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an ordered basis for a vector space V, and $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of v relative to the basis** S. The vector $(c_1, c_2, \ldots, c_n)$ in R^nconstructed from these coordinates is called the **coordinate vector of v relative to the basis** S; it is denoted by $(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$.

Example: Consider $\mathbf{v} = (-1, 7, 2) \in R^3$. The set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2, -1, -1)$,$\mathbf{v}_2 = (-2, 1, 2)$, and $\mathbf{v}_3 = (3, 5, 4)$ forms a basis for R^3 (verify). Find $(\mathbf{v})_S$ (where S is the standard basis for R^3) and $(\mathbf{v})_B$.

Suppose $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. We want c_1, c_2, c_3.

$$\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} c_1 + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} c_2 + \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} c_3 = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & 5 \\ -1 & 2 & 4 \end{bmatrix}}_{B} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}}_{\vec{v}}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & -1 \\ -1 & 1 & 5 & 7 \\ -1 & 2 & 4 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So $\vec{v} = -6\vec{v}_1 - 4\vec{v}_2 + 1\vec{v}_3$

OR $-6(2,-1,-1) - 4(-2,1,2) + 1(3,5,4) = (-1,7,2)$.

$\boxed{(\vec{v})_B = (-6,-4,1)}$ This is the coordinate vector of $\vec{v}$ relative to B.

Now consider the standard basis $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for R^3.

$$(-1,7,2) = -1\,\vec{e}_1 + 7\vec{e}_2 + 2\vec{e}_3$$
$$= -1(1,0,0) + 7(0,1,0) + 2(0,0,1)$$

So $\boxed{(\vec{v})_S = (-1,7,2)}$

$$1\begin{bmatrix}1&0\\1&0\end{bmatrix}+2\begin{bmatrix}1&1\\0&0\end{bmatrix}+3\begin{bmatrix}1&0\\0&1\end{bmatrix}+4\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}6&2\\5&3\end{bmatrix}$$

#16 First show that the set $S = \{A_1,\ A_2,\ A_3,\ A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

$$A_1 = \begin{bmatrix}1&0\\1&0\end{bmatrix},\ A_2 = \begin{bmatrix}1&1\\0&0\end{bmatrix},\ A_3 = \begin{bmatrix}1&0\\0&1\end{bmatrix},\ A_4 = \begin{bmatrix}0&0\\1&0\end{bmatrix};\ A = \begin{bmatrix}6&2\\5&3\end{bmatrix}$$

Show S is a basis:

$$\begin{bmatrix}1&1&1&0\\0&1&0&0\\1&0&0&1\\0&0&1&0\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

So it's a basis.

$$c_1A_1+c_2A_2+c_3A_3+c_4A_4=A$$

$c_2=2$, $c_3=3$ $\quad c_1+c_2+c_3=6 \Rightarrow c_1=1$

$c_4=4$ so $1A_1+2A_2+3A_3+4A_4=A$

$$(A)_S=(1,2,3,4)$$

Show that

$W = \{(x,y) \mid 2x+3y=0\}$ is a subspace of $\mathbb{R}^2$.

Let $\underline{\vec{u}=(u_1,u_2)}$, $\underline{\vec{v}=(v_1,v_2)\in W}$.

(That is, $\underline{2u_1+3u_2=0}$ and $\underline{2v_1+3v_2=0}$.)

$\vec{u}+\vec{v}=(u_1+v_1,\ u_2+v_2)$

$$\begin{aligned} 2(u_1+v_1)+3(u_2+v_2) &= 2u_1+2v_1+3u_2+3v_2 \\ &= \underline{2u_1+3u_2}+\underline{2v_1+3v_2} \\ &= 0+0=0 \end{aligned}$$

So $\vec{u}+\vec{v}\in W$.

Let $k\in\mathbb{R}$. Then $k\vec{u}=(ku_1, ku_2)$

So $2(ku_1)+3(ku_2)=k(2u_1+3u_2)=k(0)=0$

So $k\vec{u}\in W$.

The set is a subspace.

Upper Triangular: $U = \begin{bmatrix} d_1 & & \# \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix}$

Let $U = [u_{ij}]$ such that $u_{ij} = 0$ if $i > j$.

$$S = \left\{ B \in M_{22} \;\middle|\; AB = BA \text{ for } A = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} \right\}$$

Let $B, C \in S$. That is, $BA = AB$ and $CA = AC$.